DOI: 10.2478/auom-2025-0005
An. Şt. Univ. Ovidius Constanţa

sciendo
Vol. 33(1),2025, 77–92

Coefficient bounds for q-convex functions related to q-Bernoulli numbers

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Abstract

The main objective of this paper is to present and investigate a subclass $\mathcal{C}(b,q)$ of q-convex functions in the unit disk that is defined by the q-Bernoulli numbers. For this subclass, we find the upper bounds on the Fekete-Szeg functional, the coefficient bounds, and the second Hankel determinant.

1 Introduction and definitions

Let $\mathcal{A}$ denote the family of functions l analytic in the open unit disk $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$ of the form

$$l(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

Denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ containing all univalent functions in $\mathcal{U}$.

The class of starlike functions in $\mathcal{U}$ will be denoted by $\mathcal{S}^*$, which consists of normalized functions $l \in \mathcal{S}$ that satisfy the following conditions:

$$\mathcal{S}^* = \left\{ l \in \mathcal{S} : \Re\left(\frac{zl'(z)}{l(z)}\right) > 0;\ z \in \mathcal{U} \right\}.$$

Key Words: Analytic and univalent functions, q-derivative, q-convex functions, q-Bernoulli numbers, Fekete-Szeg inequality, Hankel determinant.
2010 Mathematics Subject Classification: Primary 30C45, 30C50; Secondary 30C80.
Received: 07.01.2024
Accepted: 30.04.2024

The class of convex functions in $\mathcal{U}$ will be denoted by $\mathcal{C}$, which consists of normalized functions $l \in \mathcal{S}$ that satisfy the following conditions:

$$\mathcal{C} = \left\{ l \in \mathcal{S} : \Re\left(1 + \frac{zl''(z)}{l'(z)}\right) > 0;\ z \in \mathcal{U} \right\}.$$

Let l and g be analytic functions in $\mathcal{U}$. We define that the function l is subordinate to g in $\mathcal{U}$ and denoted by

$$l(z) \prec g(z) \qquad (z \in \mathcal{U}),$$

if there exists a Schwarz function ω, analytic in $\mathcal{U}$ with $\omega(0) = 0$ and $|\omega(z)| < 1$ such that $l(z) = g(\omega(z))$ in the unit disk $\mathcal{U}$.

If the value of g in $\mathcal{U}$ is a univalent function, then

$$l(z) \prec g(z) \Longleftrightarrow l(0) = g(0) \quad \text{and} \quad l(\mathcal{U}) \subset g(\mathcal{U}).$$

Let $\mathcal{P}$ be the class of analytic functions p in $\mathcal{U}$ with $p(0) = 1$ and $\Re(p(z)) > 0$ such that $p \in \mathcal{P}$ if and only if $p(z) \prec (1+z)/(1-z)$.

The following lemmas will be necessary in order to establish our main results.

Lemma 1.1. *[20] Let $p \in \mathcal{P}$ with $p(z) = 1 + c_1 z + c_2 z^2 + \cdots$, then*

$$|c_n| \leq 2 \quad (n \in \mathbb{N} = 1, 2, \ldots) \tag{1.2}$$

and the inequality is sharp.

Lemma 1.2. *[8] Let $p \in \mathcal{P}$ with $p(z) = 1 + c_1 z + c_2 z^2 + \cdots$, then*

$$2c_2 = c_1^2 + x(4 - c_1^2) \tag{1.3}$$

and

$$4c_3 = c_1^3 + 2(4 - c_1^2)c_1 x - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2)\left(1 - |x|^2\right)z, \tag{1.4}$$

for some x, z with $|x| \leq 1$ and $|z| \leq 1$.

Hankel Determinants are an important tool in the theory of univalent functions. They can be used, for instance, to demonstrate the rationality of a function of bounded characteristic in $\mathcal{U}$, or a function that is the ratio of two bounded analytic functions with integral coefficients in its Laurent series around the origin [4]. The Hankel determinants [18] $H_j(n)$ $(n = 1, 2, \ldots,\ j = 1, 2, \ldots)$ of the function l are defined by

$$H_j(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+j-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+j} \\ \vdots & \vdots & & \vdots \\ a_{n+j-1} & a_{n+j} & \cdots & a_{n+2j-2} \end{vmatrix} \qquad (a_1 = 1).$$

Fekete-Szeg functional is the name given to the functional $H_2(1) = a_3 - a_2^2$, and the second Hankel determinant is the name given to the functional $H_2(2) = a_2a_4 - a_3^2$.

Definition 1.1. *Let $q \in (0,1)$. The q-derivative (or q-difference) operator, introduced by Jackson [10], [11], is defined as*

$$D_q l(z) = \begin{cases} \frac{l(z)-l(qz)}{(1-q)z}, & \text{if} \quad z \neq 0 \\ l'(0), & \text{if} \quad z = 0 \end{cases}. \tag{1.5}$$

We note that

$$\lim_{q\to 1} D_q l(z) = l'(z)$$

if l is differentiable at z. From (1.5), we decude that for function $l \in \mathcal{A}$

$$D_q l(z) = 1 + \sum_{n=2}^{\infty} [n]_q \, a_n z^{n-1}, \tag{1.6}$$

where $[n]_q$ is given by

$$[n]_q = \frac{1-q^n}{1-q}, \quad [0]_q = 0 \tag{1.7}$$

and the q-factorial is given by

$$[n]_q! = \begin{cases} 1, & n = 0 \\ \prod_{k=1}^{n} [k]_q\,, & n \in \mathbb{N} \end{cases}. \tag{1.8}$$

As $q \to 1-$, we obtain $[n]_q \to n$. If we choose the function $h(z) = z^n$, while $q \to 1-$, we can thus have

$$D_q h(z) = D_q z^n = [n]_q \, z^{n-1} = h'(z),$$

where the ordinary derivative is denoted by h'.

Definition 1.2. *([1]) A function $l \in \mathcal{S}$ is said to be in the $\mathcal{C}_q$ such that*

$$\mathcal{C}_q = \left\{ l \in \mathcal{S} : \Re\left(\frac{D_q\left(zD_q l(z)\right)}{D_q l(z)} \right) > 0;\ q \in (0,1),\ z \in \mathcal{U} \right\}. \tag{1.9}$$

It is clear that $\lim\limits_{q\to 1-} \mathcal{C}_q = \mathcal{C}$.

q-Bernoulli numbers and q-Bernoulli polynomials possess many interesting properties and arise in many areas of physics and mathematics. In 1948, Carlitz [5] introduced the q-Bernoulli numbers and polynomials. Many mathematicians have studied q-Bernoulli numbers and q-Bernoulli polynomials (see [3], [7], [12], [13], [17], [21]). Srivastava [24] give several remarkably shorter proofs of each of the Euler polynomials and classical Bernoulli were expressed as finite sums involving the Hurwitz zeta function. The q-Bernoulli numbers and polynomials and, in a modified notation the q-Stirling numbers of the second kind introduced there are studied by Choi et al. [6]. Srivastava ([25]) give a brief expository and historial account of the various basic (or q-) extensions of the classical Bernoulli numbers and polynomials.

According to Jackson's q-exponential functions, the generating function for q-Bernoulli polynomials is defined as follows [17]:

$$\begin{aligned} F_x(z) &= \frac{ze_q(xz)}{E_q\left(\frac{z}{2}\right)\left(e_q(\frac{z}{2})-e_q(-\frac{z}{2})\right)} \\ &= \frac{ze_q(xz)e_q(-\frac{z}{2})}{e_q(\frac{z}{2})-e_q(-\frac{z}{2})} \\ &= \sum_{n=0}^{\infty} B_n^q(x)\frac{z^n}{[n]!} \end{aligned} \tag{1.10}$$

Jackson's q-exponential functions are found here [12]

$$e_q(x)=\sum_{n=0}^{\infty}\frac{x^n}{[n]_q!}, \qquad E_q(x)=\sum_{n=0}^{\infty}q^{\frac{n(n-1)}{2}}\frac{x^n}{[n]_q!}. \tag{1.11}$$

The following formula can be used to relate two q-exponential functions

$$e_q(x)E_q(-x)=1.$$

To obtain the recursion formula, one can simply q-differentiate the generating function with respect to x

$$D_q^x B_n^q(x)=[n]_q\, B_{n-1}^q(x),$$

where $B_0^q(x)=1$.

The q-Bernoulli numbers are $b_n^q\equiv B_n^q(0)$ for $n\geq 0$.

As per the definition given above, the generating function of q-Bernoulli numbers is given by

$$F_0(z)=\frac{z}{E_q\left(\frac{z}{2}\right)\left(e_q(\frac{z}{2})-e_q(-\frac{z}{2})\right)}=\sum_{n=0}^{\infty}b_n^q\frac{z^n}{[n]!}. \tag{1.12}$$

Obviously, b_n^q is for some values of n as following:

$$\begin{aligned} b_0^q &= 1, \\ b_1^q &= -\frac{1}{2}, \\ b_2^q &= \frac{1}{4}\left([2]-\frac{1}{[3]}-q\right)=\frac{q+q^2}{4\left(1+q+q^2\right)}, \\ b_3^q &= 0. \end{aligned} \tag{1.13}$$

Here, we present a new subclass of q-convex functions in $\mathcal{U}$ that belong to the class $\mathcal{C}_q$ and are associated with q-Bernoulli numbers.

Definition 1.3. *A function $l \in \mathcal{S}$ is regarded to be in the function class $\mathcal{C}(b,q)$ if it meets the conditions given below:*

$$\frac{D_q\left(zD_q l(z)\right)}{D_q l(z)} \prec F_0(z)=\frac{z}{E_q\left(\frac{z}{2}\right)\left(e_q(\frac{z}{2})-e_q(-\frac{z}{2})\right)} \qquad (z\in\mathcal{U}) \tag{1.14}$$

for $q \in (0,1)$.

First used by Srivastava [23] in Geometric Function Theory, the basic (or q-) hypergeometric functions were also introduced by [9] q-extension of the class of starlike functions via the q-derivative operator D_q. The q-calculus was really used with a solid basis for the application of Geometric Function Theory. Afterwards, numerous mathematicians have produced a substantial amount of work, which has been fundamental to the advancement of geometric function theory. Coefficient bounds of classes of q-starlike and q-convex functions were obtained by Aldweby and Darus [2]. Coefficent estimates of the subclasses of q-starlike and q-convex functions of complex order were studied by Seoudy and Aouf [22]. The coefficient inequality for q-starlike functions was obtained by Uar [32]. The coefficient inequality for q-convex and q-close-to-convex functions was obtained by Ahuja *et al.* [1]. Polatoğlu [19] examined generalized q-starlike functions growth and distortion theorems. Some new subfamilies of starlike functions were systematically defined and studied by Wongsaijai and Sukantamala [33]. Successfully expanding on the work of Wongsaijai and Sukantamala [33], Srivastava et al. [26, 27] introduced the generalized subfamilies of q-starlike functions related with the Janowski functions. The class of q-starlike functions in the conic region was investigated by Mahmood et al. [15]. Researchers Mahmood et al. [14] and Srivastava *et al.* [28] examined the class of q-starlike functions connected to Janowski functions. Fekete-Szeg inequalities for q-starlike and q-convex functions involvıng q-analogue of Ruscheweyh-type differential operator was obtained by Soni *et al.* [31]. The coefficient inequality for q-starlike functions was obtained by ağlar *et al.* [7]. Mahmood et al. [16]. investigated the upper bound of the third Hankel determinant for the class of q-starlike functions. The Hankel and Toeplitz determinants of a subclass of q-starlike functions were recently studied by Srivastava et al. [29], and the upper bound for a subclass of q-starlike functions related to the q-exponential function was studied by Srivastava et al. [30]. In present paper, using the principles of subordination, the estimates for the coefficients $|a_2|$, $|a_3|$, $|a_3 - a_2^2|$ and $|a_2a_4 - a_3^2|$ of the functions of the form (1.1) in the class $\mathcal{C}(b,q)$ have been obtained.

2 Main Results

First, we solve the coefficient inequalities for the class $\mathcal{C}(b,q)$.

Theorem 2.1. *Let $l(z)$ given by (1.1) be in the class $\mathcal{C}(b,q)$. Then, for $q \in (0,1)$,*

$$|a_2| \leq \frac{1}{2(q+1)} \tag{2.1}$$

and

$$|a_3| \leq \frac{1}{2(q+1)\left(q^2+q+1\right)} + \frac{3q^2+4q+3}{4(q+1)\left(q^2+q+1\right)^2}. \tag{2.2}$$

Proof. Since $l \in \mathcal{C}(b,q)$, there exists an analytic function ω with $\omega(0) = 0$ and $|\omega(z)| < 1$ in $\mathcal{U}$ such that

$$\frac{D_q\left(zD_q l(z)\right)}{D_q l(z)} \prec F_0(\omega(z)) \qquad (z \in \mathcal{U}). \tag{2.3}$$

Define the functions p by

$$p(z) = \frac{1+\omega(z)}{1-\omega(z)} = 1 + c_1 z + c_2 z^2 + \cdots \qquad (z \in \mathcal{U}) \tag{2.4}$$

or equivalently,

$$\begin{aligned} \omega(z) &= \frac{p(z)-1}{p(z)+1} \\ &= \frac{c_1}{2}z + \frac{1}{2}\left(c_2 - \frac{c_1^2}{2}\right)z^2 + \frac{1}{2}\left(c_3 - c_1c_2 + \frac{c_1^3}{4}\right)z^3 + \cdots \end{aligned} \tag{2.5}$$

in $\mathcal{U}$. Then p is analytic in $\mathcal{U}$ with $p(0) = 1$ and $\Re p(z) > 0$.

By using (2.5) together with $F_0(\omega(z))$ and (1.13), it is evident that

$$\begin{aligned} F_0(\omega(z)) &= \sum_{n=0}^{\infty} b_n^q \frac{(\omega(z))^n}{[n]!} \\ &= 1 - \frac{c_1 z}{4} - \left[\frac{c_2}{4} - \frac{\left(2q^2+3q+2\right)c_1^2}{16\left(q^2+q+1\right)}\right] z^2 \\ &\quad - \left[\frac{c_3}{4} - \frac{\left(2q^2+3q+2\right)c_1c_2}{8\left(q^2+q+1\right)} + \frac{\left(q+1\right)^2 c_1^3}{16\left(q^2+q+1\right)}\right] z^3 - \cdots \end{aligned} \tag{2.6}$$

Since

$$\begin{aligned} \frac{D_q\left(zD_q l(z)\right)}{D_q l(z)} &= 1 + (q+1)\,a_2 z + \left[(q+1)\left(q^2+q+1\right)a_3 - (q+1)^2 a_2^2\right] z^2 \\ &\quad + \left[\left(q^3+q^2+q+1\right)\left(q^2+q+1\right)a_4 - (q+1)\,(q+2)\left(q^2+q+1\right)a_2a_3 \right. \\ &\quad \left. + (q+1)^3 a_2^3\right] z^3 + \cdots, \end{aligned} \tag{2.7}$$

comparing coefficients in (2.6) and (2.7), we have

$$a_2 = -\frac{c_1}{4\,(q+1)}, \tag{2.8}$$

$$a_3 = -\frac{c_2}{4(q+1)\left(q^2+q+1\right)} + \frac{\left(3q^2+4q+3\right)c_1^2}{16(q+1)\left(q^2+q+1\right)^2}, \tag{2.9}$$

$$\begin{aligned} a_4 \;=\; & -\frac{c_3}{4\left(q^2+q+1\right)\left(q^3+q^2+q+1\right)} + \frac{\left(5q^3+13q^2+11q+4\right)c_1c_2}{16(q+1)\left(q^2+q+1\right)^2\left(q^3+q^2+q+1\right)} \\ & - \frac{\left(6q^3+20q^2+21q+9\right)c_1^3}{64(q+1)\left(q^2+q+1\right)^2\left(q^3+q^2+q+1\right)}. \end{aligned} \tag{2.10}$$

The bounds $|a_2|$ and $|a_3|$ can be obtained by using the triangle inequality and the well-known result $|c_n| \leq 2$ for the class $\mathcal{P}$ as follows:

$$|a_2| \leq \frac{1}{2(q+1)},$$

and

$$|a_3| \leq \frac{1}{2(q+1)\left(q^2+q+1\right)} + \frac{3q^2+4q+3}{4(q+1)\left(q^2+q+1\right)^2}.$$

This completes the proof of the Theorem 2.1. □

In the limiting case $q \to 1-$, Theorem 2.1 readily yields the following coefficient estimates.

Corollary 2.2. *Let $l(z)$ given by (1.1) be in the class $\mathcal{C}_b$. Then,*

$$|a_2| \leq \frac{1}{4}$$

and

$$|a_3| \leq \frac{2}{9}.$$

These results are sharp.

Our second main finding is Fekete-Szeg inequality, or the following properties of the class $\mathcal{C}(b,q)$.

Theorem 2.3. *Let $l(z)$ given by (1.1) be in the class $\mathcal{C}(b,q)$. Then, for $q \in (0,1)$,*

$$\left|a_3 - a_2^2\right| \leq \frac{1}{2(q+1)(q^2+q+1)}. \tag{2.11}$$

Proof. Substituting the values of (2.8) and (2.9) into the Fekete-Szeg inequality $\left|a_3 - a_2^2\right|$ for the function $\mathcal{C}(b,q)$, we get

$$\left|a_3 - a_2^2\right| = \left|\frac{c_2}{4(q+1)(q^2+q+1)} - \frac{(-q^4+q^3+4q^2+5q+2)c_1^2}{16(q+1)^2(q^2+q+1)^2}\right|.$$

By applying Lemma 1.2, we find that

$$\left|a_3 - a_2^2\right| = \left|\frac{x(4-c_1^2)}{8(q+1)(q^2+q+1)} + \frac{(q^4+q^3-q)c_1^2}{16(q+1)^2(q^2+q+1)^2}\right|.$$

Suppose now that $c_1 = c \in [0,2]$. Then an application of the triangle inequality gives

$$\left|a_3 - a_2^2\right| \leq \frac{(4-c^2)t}{8(q+1)(q^2+q+1)} + \frac{(q^4+q^3-q)c^2}{16(q+1)^2(q^2+q+1)^2}$$

with $t = |x| \leq 1$. Moreover, if we set

$$\Psi_q(c,t) = \frac{(4-c^2)t}{8(q+1)(q^2+q+1)} + \frac{(q^4+q^3-q)c^2}{16(q+1)^2(q^2+q+1)^2},$$

then, we obtain

$$\frac{\partial \Psi_q}{\partial t} = \frac{4-c^2}{8(q+1)(q^2+q+1)} \geq 0,$$

which shows that $\Psi_q(c,t)$ is an increasing function on the closed interval $[0,1]$ about t. Therefore, the function $\Psi_q(c,t)$ takes the maximum value at $t=1$, that is,

$$\max_{0\le t\le 1}\{\Psi_q(c,t)\}=\Psi_q(c,1)=\frac{4-c^2}{8(q+1)(q^2+q+1)}+\frac{(q^4+q^3-q)c^2}{16(q+1)^2(q^2+q+1)^2}.$$

We next put

$$\begin{aligned}\Phi_q(c) &= \frac{4-c^2}{8(q+1)(q^2+q+1)}+\frac{(q^4+q^3-q)c^2}{16(q+1)^2(q^2+q+1)^2}\\ &= \frac{1}{2(q+1)(q^2+q+1)}+\frac{(q^4-q^3-4q^2-5q-2)c^2}{16(q+1)^2(q^2+q+1)^2}.\end{aligned}$$

We then easily find that, at $c=0$, the function $\Phi_q(c)$ has a maximum value, which is given by

$$\left|a_3-a_2^2\right|\le\Phi_q(0)=\frac{1}{2(q+1)(q^2+q+1)}.$$

This completes the proof of Theorem 2.3. □

In the limiting case $q\to 1-$, Theorem 2.3 readily yields the following corollary.

Corollary 2.4. *Let $l(z)$ given by (1.1) be in the class $\mathcal{C}_b$. Then,*

$$\left|a_3-a_2^2\right|\le\frac{1}{12}.$$

This result is sharp.

Finally, the second Hankel determinant result for the class $\mathcal{C}(b,q)$ is as follows:

Theorem 2.5. *Let $l(z)$ given by (1.1) be in the class $\mathcal{C}(b,q)$. Then, for $q\in(0,1)$,*

$$\left|a_2a_4-a_3^2\right|\le\frac{2q^2+3q+2}{4q^2(q+1)^2(q^2+q+1)}. \tag{2.12}$$

Proof. From (2.8), (2.9) and (2.10), we have

$$\begin{aligned}\left|a_2a_4-a_3^2\right| = &\left|\frac{c_1c_3}{16(q+1)(q^2+q+1)(q^3+q^2+q+1)}-\frac{c_2^2}{16(q+1)^2(q^2+q+1)^2}\right.\\ &+\frac{\left(q^5-4q^4-9q^3-8q^2-q+2\right)c_1^2c_2}{64(q+1)^2(q^2+q+1)^3(q^3+q^2+q+1)}\\ &\left.-\frac{\left(4q^7+10q^6+15q^5+16q^4+24q^3+18q^2+9q+5\right)c_1^4}{256(q+1)^2(q^2+q+1)^4(q^3+q^2+q+1)}\right|,\end{aligned}$$

which, upon substituting for c_2 and c_3 by using Lemma 1.2, yields

$$
\begin{aligned}
\left|a_2a_4-a_3^2\right| &= \\
&\left|\frac{\left(q^5-q^3+3q+2\right)\left(4-c_1^2\right)c_1^2x}{128(q+1)^2(q^2+q+1)^3(q^3+q^2+q+1)}-\frac{\left(4-c_1^2\right)c_1^2x^2}{64(q+1)(q^2+q+1)(q^3+q^2+q+1)}\right. \\
&-\frac{\left(4-c_1^2\right)^2x^2}{64(q+1)^2(q^3+q^2+q+1)}+\frac{\left(4-c_1^2\right)\left(1-|x|^2\right)c_1z}{32(q+1)(q^2+q+1)(q^3+q^2+q+1)} \\
&\left.-\frac{\left(2q^7+12q^6+27q^5+42q^4+49q^3+20q^2+3q+1\right)c_1^4}{256(q+1)^2(q^2+q+1)^4(q^3+q^2+q+1)}\right|. \qquad (2.13)
\end{aligned}
$$

We let $c_1=c$ and assume also without restriction that $c\in[0,2]$. Then, by applying the triangle inequality on (2.13) with $|x|=t\in[0,1]$, we obtain

$$
\begin{aligned}
\left|a_2a_4-a_3^2\right| &\leq \\
&\frac{\left(q^5-q^3+3q+2\right)\left(4-c^2\right)c^2t}{128(q+1)^2(q^2+q+1)^3(q^3+q^2+q+1)}+\frac{\left(4-c^2\right)c^2t^2}{64(q+1)(q^2+q+1)(q^3+q^2+q+1)} \\
&+\frac{\left(4-c^2\right)^2t^2}{64(q+1)^2(q^3+q^2+q+1)}+\frac{4-c^2}{32(q+1)(q^2+q+1)(q^3+q^2+q+1)} \\
&+\frac{\left(2q^7+12q^6+27q^5+42q^4+49q^3+20q^2+3q+1\right)c^4}{256(q+1)^2(q^2+q+1)^4(q^3+q^2+q+1)}.
\end{aligned}
$$

We now assume that

$$
\begin{aligned}
M_q(c,t) &= \\
&\frac{\left(q^5-q^3+3q+2\right)\left(4-c^2\right)c^2t}{128(q+1)^2(q^2+q+1)^3(q^3+q^2+q+1)}+\frac{\left(4-c^2\right)c^2t^2}{64(q+1)(q^2+q+1)(q^3+q^2+q+1)} \\
&+\frac{\left(4-c^2\right)^2t^2}{64(q+1)^2(q^3+q^2+q+1)}+\frac{4-c^2}{32(q+1)(q^2+q+1)(q^3+q^2+q+1)} \\
&+\frac{\left(2q^7+12q^6+27q^5+42q^4+49q^3+20q^2+3q+1\right)c^4}{256(q+1)^2(q^2+q+1)^4(q^3+q^2+q+1)},
\end{aligned}
$$

which, upon partially differentiating with respect to t, yields

$$
\begin{aligned}
\frac{\partial M_q}{\partial t} &= \frac{\left(q^5-q^3+3q+2\right)\left(4-c^2\right)c^2}{128(q+1)^2(q^2+q+1)^3(q^3+q^2+q+1)} \\
&\quad+\frac{\left(4-c^2\right)c^2t}{32(q+1)(q^2+q+1)(q^3+q^2+q+1)}+\frac{\left(4-c^2\right)^2t}{32(q+1)^2(q^3+q^2+q+1)} \\
&> 0,
\end{aligned}
$$

which, in turn, implies that $M_q(c,t)$ increases on the closed interval $[0,1]$ about t. That is, $M_q(c,t)$ has a maximum value at $t=1$, which is given by

$$\begin{aligned}\max_{0\leq t\leq 1}\{M_q(c,t)\} &= M_q(c,1)=\frac{\left(3q^5+6q^6+9q^3+10q^2+9q+4\right)\left(4-c^2\right)c^2}{128(q+1)^2(q^2+q+1)^3(q^3+q^2+q+1)}\\ &+\frac{\left(4-c^2\right)^2}{64(q+1)^2(q^3+q^2+q+1)}+\frac{4-c^2}{32(q+1)(q^2+q+1)(q^3+q^2+q+1)}\\ &+\frac{\left(2q^7+12q^6+27q^5+42q^4+49q^3+20q^2+3q+1\right)c^4}{256(q+1)^2(q^2+q+1)^4(q^3+q^2+q+1)}.\end{aligned}$$

Also, upon setting

$$\begin{aligned}N_q(c) &= \frac{\left(3q^5+6q^6+9q^3+10q^2+9q+4\right)\left(4-c^2\right)c^2}{128(q+1)^2(q^2+q+1)^3(q^3+q^2+q+1)}\\ &+\frac{\left(4-c^2\right)^2}{64(q+1)^2(q^3+q^2+q+1)}\\ &+\frac{4-c^2}{32(q+1)(q^2+q+1)(q^3+q^2+q+1)}\\ &+\frac{\left(2q^7+12q^6+27q^5+42q^4+49q^3+20q^2+3q+1\right)c^4}{256(q+1)^2(q^2+q+1)^4(q^3+q^2+q+1)},\end{aligned}$$

we have

$$\begin{aligned}N_q'(c) &= \frac{\left(3q^5+6q^6+9q^3+10q^2+9q+4\right)\left(4-c^2\right)c}{64(q+1)^2(q^2+q+1)^3(q^3+q^2+q+1)}\\ &-\frac{\left(3q^5+6q^6+9q^3+10q^2+9q+4\right)c^3}{64(q+1)^2(q^2+q+1)^3(q^3+q^2+q+1)}\\ &-\frac{\left(4-c^2\right)c}{16(q+1)^2(q^3+q^2+q+1)}-\frac{c}{16(q+1)(q^2+q+1)(q^3+q^2+q+1)}\\ &+\frac{\left(2q^7+12q^6+27q^5+42q^4+49q^3+20q^2+3q+1\right)c^3}{64(q+1)^2(q^2+q+1)^4(q^3+q^2+q+1)}.\end{aligned}$$

If we set $N_q'(c)=0$, then $c=0$ is a root of this equation. After a suitable calculation, we can deduce that

$$\begin{aligned} N_q''(c) &= \frac{\left(3q^5+6q^6+9q^3+10q^2+9q+4\right)\left(4-c^2\right)}{64(q+1)^2(q^2+q+1)^3(q^3+q^2+q+1)} \\ &\quad -\frac{5\left(3q^5+6q^6+9q^3+10q^2+9q+4\right)c^2}{64(q+1)^2(q^2+q+1)^3(q^3+q^2+q+1)} \\ &\quad -\frac{4-c^2}{16(q+1)^2(q^3+q^2+q+1)}+\frac{c^2}{8(q+1)^2(q^3+q^2+q+1)} \\ &\quad -\frac{1}{16(q+1)(q^2+q+1)(q^3+q^2+q+1)} \\ &\quad +\frac{3\left(2q^7+12q^6+27q^5+42q^4+49q^3+20q^2+3q+1\right)c^2}{64(q+1)^2(q^2+q+1)^4(q^3+q^2+q+1)} \\ &\leq 0, \end{aligned}$$

implying that the function $N_q(c) = 0$ can take on its maximum value at $c = 0$, which is given by

$$\left|a_2a_4 - a_3^2\right| \leq N_q(0) = \frac{2q^2+3q+3}{8(q+1)^2(q^2+q+1)(q^3+q^2+q+1)}.$$

This completes the proof of Theorem 2.5. □

In the limiting case $q \to 1-$, Theorem 2.5 readily yields the following corollary.

Corollary 2.6. *Let $l(z)$ given by (1.1) be in the class $\mathcal{C}_b$. Then,*

$$\left|a_2a_4 - a_3^2\right| \leq \frac{1}{24}.$$

This result is sharp.

3 Conclusion

In the open unit disk $\mathcal{U}$, we have introduced a new subclass $\mathcal{C}(b, q)$ of q-convex functions that are subordinate to the q-Bernoulli function by using the basic or q-calculus. For this subclass of q-convex functions related to the q-Bernoulli function, we have successfully derived the upper bounds of the Fekete-Szeg functional and the second Hankel determinant.

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